Searchability of networks

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We investigate the searchability of complex systems in terms of their interconnectedness. Associating searchability with the number and size of branch points along the paths between the nodes, we find that scale-free networks are relatively difficult to search, and thus that the abundance of scale-free networks in nature and society may reflect an attempt to protect local areas in a highly interconnected network from nonrelated communication. In fact, starting from a random node, real-world networks with higher order organization like modular or hierarchical structure are even more difficult to navigate than random scale-free networks. The searchability at the node level opens the possibility for a generalized hierarchy measure that captures both the hierarchy in the usual terms of trees as in military structures, and the intrinsic hierarchical nature of topological hierarchies for scale-free networks as in the Internet.

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I. INTRODUCTION

Each element interacts directly only with a few particular elements in most complex systems. Distant parts of the network thereby formed can consequently communicate through sequences of local interactions. In this way all parts of the network can be reached from other parts, but not all such communications are equally easy or accurate [1–4]. The purpose of this paper is to investigate the interplay between searchability of a network and the network structure. By searchability or navigability we mean the difficulty of sending a signal between two nodes in a network without disturbing the remaining network. We use a city-street network to illustrate the concept of navigability in networks [5]. As in Fig. 1(c) the streets are identified as nodes and intersections between the streets as links between the nodes. From this point of view, the above statement reads: A pedestrian or driver on a street in a city, can by multiple choices reach any other street in the city via the intersections. However, not all streets are as easy to find, and the difficulty of finding a street may vary from city to city.

In the current paper we investigate how different network topologies influence the average amount of information that is needed to send a signal from one node to another node in the network. We consistently concentrate on specific signaling, and focus only on locating one specific node without disturbing the remaining network. This is different from the nonspecific broadcasting where any input is amplified by all exit links of every node along all paths, as in spreading of spam or propagation of diseases and computer viruses [6,7]. We present a quantification of the specific signaling and justify our choice of measure by its minimum information property.

II. SEARCH INFORMATION

We consider a specific signal, or a walker, on a network and assume that the specific signal from a source s to a target t is a signal that travels along the shortest path, and thereby minimizes the disturbance on other nodes. This assumption is made on the basis that the shortest path is a good estimate for typical traffic in a network [8]. We will later discuss the alternative model, to follow the minimal information path, not necessarily coinciding with the shortest path. The minimal amount of information needed to follow a specific short-

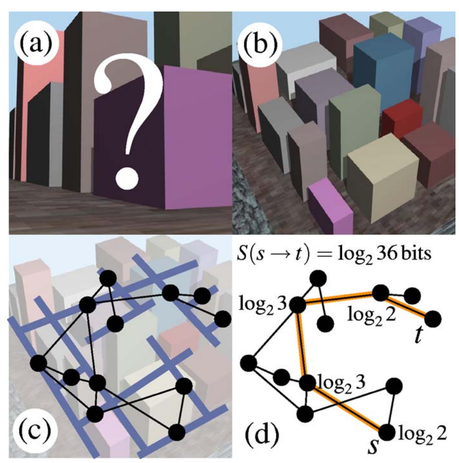

FIG. 1. (Color online) An example where the search information is an important concept. (a) illustrates a visitor's perspective of an unknown city. The visitor therefore asks a citizen with the perspective (b) of the city, or rather the higher abstraction level (c). This level is the dual map of the city, a network where streets are identified as nodes and intersections between streets as links between the nodes. We use this level to quantify the search information in (d); the average number of yes-no questions the visitor must ask the citizen to find a specific street. The necessary information to walk the shortest path from s in the lower right corner to t in the upper right corner is $\log_2 36$ bits or roughly 6 yes/no-questions.

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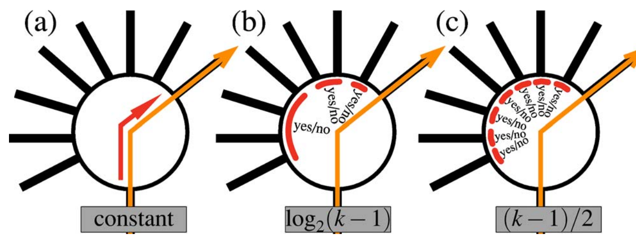


FIG. 2. (Color online) The information cost at each node depend on the ordering of the link. (a) The information cost does not depend on the degree if there is only one possible link. (b) It is possible to ask yes-no questions and successively eliminate groups of wrong exits if the links are ordered. Every yes-no question optimally reduces the number of possible links with $1/2$ and the cost is $\log_2(k-1)$ to find the correct exit link. (c) If the links are unordered such groupings are impossible and every exit link must be considered. In such a scenario the average number of necessary yes-no questions is $(k-1)/2$.

est path is determined by the degrees of the nodes along the path, i.e., the number and size of the branch points between the nodes. That is, a walker on the network first has to choose the right exit link (we call it the path link) among the k_s possible links from s. The cost depends on the available global information on the node and the way the information is organized at the node [9,10]. If no information is available, the choice must be random, and the walker will perform a random walk. We here consider scenarios where the network represents the communication backbone of a system with available information on the node level, e.g., social networks [3,4], computer networks [10], city networks [5,11], etc. In principle, if the exit links are unordered, one yes-no question must be asked for every exit link to find the path link. On average this would give rise to an average cost of $k/2$ yes-no questions or $(k-1)/2$ if the arrival link at the node is known and one link immediately can be excluded. This is illustrated in Fig. 2(c).

The other extreme situation is when the exit path somehow is given by default or the information cost can be neglected in comparison to the walk on the shortest path itself [Fig. 2(a)]. We here focus on the case where the links are ordered, like intersections along a road. In this case a question can be used to reduce the possible outcomes by a factor 2. A city example: The yes-no answer to "Does any one of the eight closest roads lead toward the station?" reduces the outcome to eight roads if it there were 16 possible intersecting roads to choose from. The total number of bits, or roughly the number of yes-no questions, necessary to find the path link is $\log_2(k)$ or $\log_2(k-1)$ if the arrival node is known to not lead to the target as in Fig. 2(b).

That is, $\log_2(k_s)$ bits of information are necessary at the start node s, where k_s is the degree of s. Subsequently the walker at each node $j \in p(s,t)$ along the path $p(s,t)$ has to choose the particular exit link along the path. Given the knowledge to follow the path to j, there are k_j-1 unknown exit links from j, and the information needed to make the next step is $\log_2(k_j-1)$. As a result the total information needed to follow the path is

$$S_u(p(s,t)) = \log_2(k_s) + \sum_{j \in p(s,t)} \log_2(k_j - 1), \tag{1}$$

where $p(s,t)$ includes nodes on the path between s and t, but not the start and end nodes s and t [see Fig. 3(a)]. We use the notation S_u to emphasize that the walk is a result of decisions for a specific and unique path and repeat that we use $\log_2(k_j-1)$ at every step but the first since the link of arrival is known (Fig. 2).

If there is more than one shortest path between s and t the information needed to travel along one of the shortest paths has to include the thereby added degenerate possibilities [1]. Degenerate paths imply that more than one exit link can lead the walker closer to the target from each node, and should be reflected in a decreased path information $S_d(\{p(s,t)\})$; the subscript d is for degenerate paths and $\{p(s,t)\}$ is for the set of paths between s and t. If a node j has k_j links, of which η_j links point toward the target node t, then the number of bits to locate one of the correct exits is reduced to $\log_2[(k_j-1)/\eta_j]$ [and to $\log_2(k_s/\eta_s)$ for the first step at the source node s]. In this definition we make the assumption that the probability of choosing any exit link on a shortest path from the current node is equal. Therefore each of the degenerate

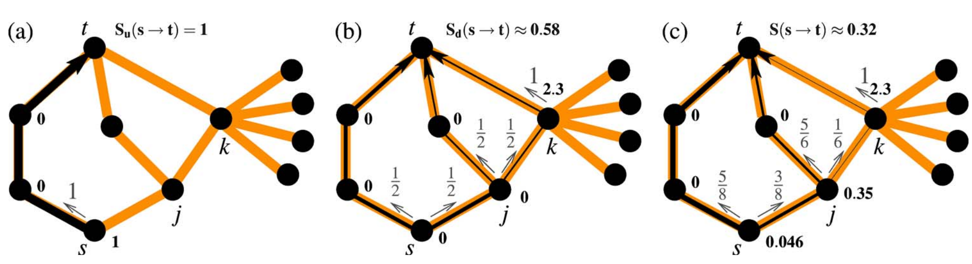


FIG. 3. (Color online) Search information with degenerate paths between the source s and target t. The numbers around the nodes indicate with what probability the link is chosen on the walk to t. The boldfaced number is the information cost in bits at the given node. The total information cost $S(s \to t)$ above every network is given by the average cost over all paths marked with black lines (the sum of the costs at the nodes along the paths) weighted with the probability to walk the path (width of black lines). We present three scenarios. (a) The walker aims to take a specific shortest path in, the cheapest informationwise. (b) The walker chooses between two exits, both on the shortest path, randomly. This results in a lower total information cost since a random choice does not cost any information, even though some walks will go through the expensive hub to the right (2.3 bits). (c) The walker chooses to minimize the average information cost between s and t. The difference between (b) and (c) is clear from the choice at node j. In (c) more information is used at this node to avoid the higher cost of going to the hub to the right. This is completely avoided in (a) by going to the left at s, but at a higher information cost.

paths will be selected with a different probability, as indicated in the example of Fig. 3(b). That is, each path in the set of degenerate paths $\{p(s,t)\}$ is selected with probability

$$P(p(s,t)) = \frac{1}{\eta_s}\prod_{j\in p(s,t)}\frac{1}{\eta_j}. \tag{2}$$

The average number of bits needed to follow a random shortest path is accordingly

$$S_d(s\to t) = \sum_{\{p(s,t)\}} P(p(s,t))\log_2\left(\frac{k_s}{\eta_s}\prod_j\frac{k_j-1}{\eta_j}\right). \tag{3}$$

This simplifies to Eq. (1) in the case where there is only one degenerate path. When there are degenerate paths between s and t, $S_d(s\to t)$ does not distinguish paths that are difficult to follow from the easier ones, but just averages.

The average path information $S_d(s\to t)$ is closely related to the earlier introduced search information [5,12–14]:

$$S(s\to t) = -\log_2\left(\sum_{\{p(s,t)\}}\frac{1}{k_s}\prod_j\frac{1}{k_j-1}\right), \tag{4}$$

where the sum runs over the set $\{p(s,t)\}$ of degenerate shortest paths between s and t. Thus, again, if there are no degenerate shortest paths, $S(s\to t)=S_u(s\to t)$. If there are degenerate paths, the relative weighting of these paths differs. In the S_d measure each path is weighted according to the branching of shortest path shown in Fig. 3(b), and is thus the typical information needed to follow a random branch of one of the shortest paths through the network. In contrast S measures the minimal information value of knowing the full path and the subscript m for minimal is omitted. S is defined as $-\log_2$ of the probability that a nonguided signal emitted from s arrives at t with minimal number of steps. For all networks we have tested S_d is maximally a few percent larger than S, reflecting that situations where one of the branches is substantially more difficult to travel only gives a small additional correction to S_d (see Fig. 4). Also, we always found indistinguishable results when we analyzed the networks in terms of the conditional uniform test $S-S$ (random) or in terms of S_d-S_d (random) [15,16].

To get the corresponding probabilities to follow a given path as in Eq. (2) we present a simple example of the minimum information property of S, and choose the path from j to t in Fig. 3(c) as an example path. Let the probability to take the left path be q_1 and the right path via the hub be $q_2=1-q_1$ and further the probability to reach the target be p_1 after the left choice is taken and p_2 if the right choice is taken. The probability to choose the link down to the left is 0, since it is not on a shortest path to t. Then the total information cost from j to t is

$$S(j\to t) = (\log_2 3 + q_1\log_2 q_1 + q_2\log_2 q_2) - (q_1\log_2 p_1 + q_2\log_2 p_2), \tag{5}$$

where the first term in parentheses on the right-hand side is the information cost to pay at node j. The full expression of this term reads

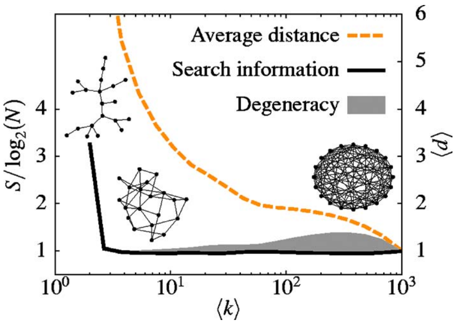


FIG. 4. (Color online) Search information of random Erdős-Rényi (ER) networks as a function of average degree $\langle k\rangle$. The number of nodes is $N=10^3$ and we keep the networks connected. The shaded area is the contribution from degenerate paths with the upper edge corresponding to the definition of S_u in Fig. 3(a). The definition in Fig. 3(b) is inseparable in this plot from the search information according to Fig. 3(c). The degenerate paths make the highly connected networks more searchable, mainly due to degenerate paths of length 2 between each pair of nodes. Thus the resulting S is lower than that obtained when considering information associated to locating just one of the shortest paths (upper border of shaded area). For very low degrees, $\langle k\rangle\sim 2$, the organization of the networks opens for a broad range of different topologies with very different searchability; the average shortest path increases and finally no degenerate paths exist.

$$\sum_{i=1}^{3} -\frac{1}{3}\log_2\left(\frac{1}{3}\right) - \sum_{i=1}^{3} -q_i\log_2(q_i), \tag{6}$$

and is the difference between the information entropy of a random choice and the information entropy of the actual choice—the meaningful information of the choice. The information cost payed at node j ensures that the walker takes the path to the left with probability q_1 and to the right with probability q_2. This is equivalent to the meaningful information content of a policeman in the crossing who points toward the left with probability q_1, to the right with probability q_2, and never down to the left (since it is not on a shortest path to t). The remaining two terms in Eq. (5) represents the cost from the next step to the target as two contributions according to Eq. (4), weighted with the probabilities q_1 and q_2 of choosing the paths.

We set $dS/dq_1=0$ to find the minimum. With $q_2=1-q_1$ we get

$$0 = \log_2 q_1 - \log_2(1-q_1) - \log_2 p_1 + \log_2 p_2, \tag{7}$$

or $q_1/q_2=p_1/p_2$, satisfied by

$$q_1 = p_1/(p_1+p_2), \quad q_2 = p_2/(p_1+p_2). \tag{8}$$

Inserting this back in Eq. (5) gives

$$S(j\to t) = -\log_2\left(\frac{1}{3}p_1 + \frac{1}{3}p_2\right), \tag{9}$$

which is identical to Eq. (4). Effectively q_1 and q_2 weight the probability of choosing an exit from j with the difficulty of following it. For example, paths that contain large hubs will be suppressed because the probability of following such paths randomly is lower.

To be able to characterize the complete network in terms of searchability we define S as the average of pairwise search information between nodes over all pairs of nodes

$$S = \frac{1}{N(N-1)} \sum_{s \neq t} S(s \to t). \tag{10}$$

Thus, although S is defined in terms of global random walkers it should be interpreted as subsequent and local minimization of information costs to navigate to a target node. Thus, it is different from the random walker approach that has been used to characterize topological features of networks [17,18], including first passage times [19], large scale modular features [20], and search utilizing topological features [9]. Neither should the search information with its logarithm of base 2 be mixed up with entropy measures associated with the degree distribution [21], measures related to the dominating eigenvector of the adjacency matrix [22], or different flows on networks like betweenness centrality and closeness centrality[23–25]. Instead S measures the amount of information that turns a random walker to a directed walker that follows a shortest path (or any other chosen path) between the source s and target t.

Some insight into the search information S, which also makes the difference from a pure entropy measure clear, is obtained if we consider the simple average along one of the shortest paths, and ignore information associated with having arrived from a link that cannot be leading closer to t:

$$\mathcal{S}(s,t) = \sum_{p(s,t)} P(p(s,t)) \log_2\left(k_s \prod_j k_j\right) \tag{11}$$

with a total average path information

$$\mathcal{S} = \frac{1}{N(N-1)} \sum_{j=1}^{N} b(j) \log_2(k_j), \tag{12}$$

which differs from a pure entropy measure of the form $\Sigma p \log p$ since $b(j)$ is proportional to k_j only when the walk is random. Here $b(j)$ is the traffic betweenness of the node j, defined as the number of shortest paths between pairs of nodes in the network that pass through node j, including paths that start at j or paths that end at j. This traffic betweenness differs from the usual betweenness [23,24] by the different treatment of degenerate paths, in the sense that a given degenerate path contributes to betweenness with a weight given by the difficulty of walking the path according to Eq. (4). In practice, in all the real networks that we have investigated, we found that the difference is negligible. We thus expect relatively large S values for networks (1) where there are many nodes on the shortest path between other nodes [most $b(i)$ large], and (2) where most traffic goes through highly connected nodes. Point 1 predicts large S for modular networks, whereas point 2 suggests relatively large S for networks with broad degree distributions. The path length is indirectly coupled to points 1 and 2; stringy networks as well as regular networks with long average path lengths have high S and starlike networks have small S despite point 2, because of the very short paths. In the remaining part of this paper we will examine the interplay between S and global topology in detail.

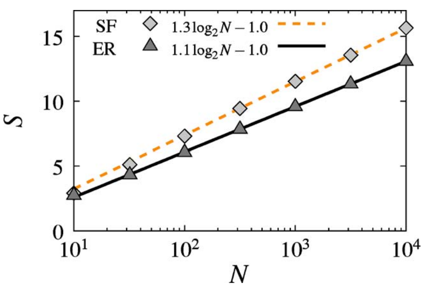


FIG. 5. (Color online) S as function of system size N for fully connected random network topologies with fixed average degree $\langle k \rangle = 5$. ER refers to Erdős-Rényi random networks and SF to random scale-free networks with degree distribution $\propto 1/(k_0+k)^{2.4}$ with k_0 adjusted for every network size so that $\langle k \rangle$ is kept fixed. A star network with one node connected to the remaining nodes in the network and the remaining links randomly distributed scales asymptotically as $S = \log_2 N + 1$ as in principle every shortest path goes through the hub with the cost $\log_2(N-1)$.

III. SEARCH INFORMATION IN MODEL NETWORKS

The search information is topology dependent, and in this section we present how S captures the average degree, the degree distribution, and higher order topological organization of the networks.

Figure 4 shows how the S depends on average degree $\langle k \rangle$ in a random network. The lower curve is the total S and the shaded area represents the contribution from degenerate paths. The upper border of the shaded area is consequently S_u, the search information without degenerate paths. S_d ($S_u \geq S_d \geq S$), that weights paths according to branch points along the paths, is within the shaded area (although it is indistinguishable from the lower curve in the present case). Notice that the figure mostly examines very high $\langle k \rangle$ values where most pairs of nodes are connected by multiple degenerate paths of length 2. This explains the reduction in search information due to degenerate paths, which becomes small for the real-world networks when $\langle k \rangle$ is 1–10. For these small $\langle k \rangle$, S depends crucially on the global topological organization: it is $\log_2 2 = 1$ for a one-dimensional string, $\log_2 N$ for a star, but of order $N/4$ for a stringy structure with many separated branches (see Fig. 4). The increase of the average shortest path length $\langle l \rangle$, plotted as a dashed line, indicates that the stringy structure dominates in the ensemble of random networks with low $\langle k \rangle$.

In Fig. 5 we demonstrate that $s = S/\log_2(N)$ is nearly a size-independent way to compare networks of different size with each other [5]. Thus this quantity is an invariant for any given type of network topology, whether it is dominated by a single hub (star), whether it is scale-free (SF), or whether it is of Erdős-Rényi (ER) type. In all cases we compare networks with the same average degree and find that s nicely differentiates between different types of networks with a given amount of links between the nodes. The asymptotic

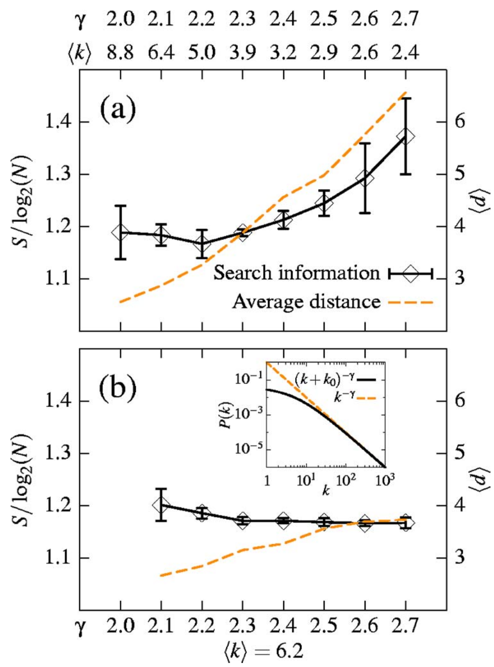


FIG. 6. (Color online) S as a function of the exponent γ for random scale-free networks with degree distribution $P(k) \propto 1/k^{\gamma}$ in (a). Varying γ implies varying average degree $\langle k \rangle$ according to the second x-label row. An increased γ also implies a decreased frequency of large hubs and a lengthened average shortest path between nodes. To separate the effects we in (b) study the distribution $P(k) \propto 1/(k_0+k)^{\gamma}$ with k_0 set to keep the $\langle k \rangle$ fixed, as in the insert. Here we see that even though the average shortest path increases, S decreases slightly due to a decreased frequency of hubs. However, compared to (a) the average shortest path increases substantially less with increasing γ, due to the constant $\langle k \rangle$. The increasing average shortest path accordingly dominates over the decreasing frequency of hubs for $\gamma > 2.2$ in (a). The size of the networks is $N = 10^4$ in both (a) and (b).

logarithmic scaling can be understood by the logarithmic increase in average shortest path length for Erdős-Rényi networks, $\langle l_{sp} \rangle \propto \log N$ [26], and constant cost at every node ($\log_2 \langle k \rangle$). For scale-free networks it is a little deeper, but simple for the extreme cases. For $\gamma = 2$ the average shortest path length is constant and the size of the largest hub in the network scales linearly with the system size [26]. As almost all shortest paths will go through this "superhub" as in a star network, the search information is proportional to $\log_2 N$. For $\gamma > 3$ the average shortest path length scales as $\log N$ and the largest hub is finite, similar to Erdős-Rényi networks [27].

From Fig. 5 we also notice that scale-free networks have the largest S, at least as long as we consider a random organization of the topology. This is because nodes with large values of k_i also have large b_i, and therefore contributes relatively more to the overall confusion according to Eq. (12). This fact is explored more in Fig. 6 where we show the variation of $S/\log_2(N)$ as function of degree distribution quantified by γ. At low $\gamma \sim 2$, where effectively a scale-free network behaves very similar to a star network, the largest hubs tend to be connected to a major fraction of the system.

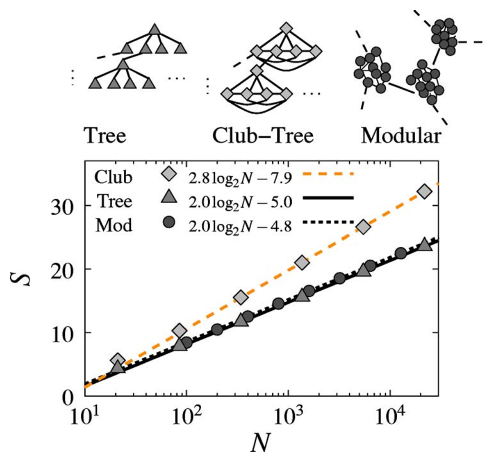


FIG. 7. (Color online) S versus N for tree, club-tree and modular networks. Both trees have a branching ratio d of 4, as seen in the illustration above. The modular network consists of communities of ten nodes, each of them connected to five other nodes. Each community is in turn connected with three other communities.

A typical path therefore passes through a major hub of degree $k \propto N$ and maybe one more node as indicated by the average shortest path length. For larger γ the high cost of passing nodes with $k \propto N$ disappears, but the total average cost nevertheless increases since the path length increases rapidly. In Fig. 6(b) the average degree is kept constant by adjusting k_0 in the degree distribution $P(k) \propto (k_0+k)^{-\gamma}$. This weakens the increase in average path length as γ increases and S instead slowly decreases because the probability for having very large hubs decreases.

We now turn to networks with narrow degree distributions, but nonrandom topologies and start with an illustrative calculation of S for a tree hierarchy. We obtain $S = 2\log(N) - 5$ numerically for trees of different branching ratios d (Fig. 7), which was corroborated analytically for a binary tree. However, S depends on addition of links to the tree, and in particular S is larger for the club tree, as numerically demonstrated in Fig. 7. In any case S for trees is much larger than for random networks. The reason why trees are perceived as efficient is (1) that they are efficient seen from the top (e.g., data structures), and (2) trees are mostly associated not to specific signaling, but rather to broadcasting of information, where everyone in a certain section is given the same information (e.g., military organization). Even higher information cost has a regular network (every node connected to twice the dimension d of the lattice) as the shortest path length scales as $N^{1/d}$. If the links of the regular network instead represents street segments between intersections in a square city like Manhattan (streets and avenues mapped to nodes and intersections to links between the streets and avenues in a fully connected bipartite network [5]), the result is completely different. Let N streets be divided into $N/2$ north-south (NS) streets, and $N/2$ east-west (EW) streets. Going from any NS street to a particular EW street demands information about which of the $N/2$ exits is correct. This information cost is $S(\text{NS} \rightarrow \text{EW}) = \log_2(N/2)$. To go from one NS street to another NS street means that any of the $N/2$ EW streets can be chosen. Each path is thus assigned a probabil-

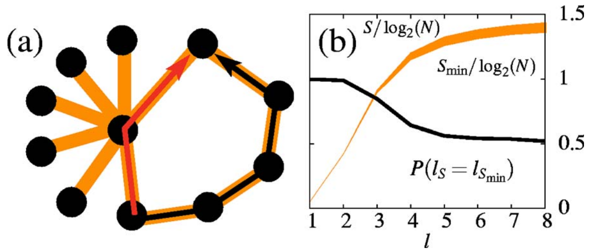


FIG. 8. (Color online) The minimum information path of length $l_{S_{min}}$ is the path between two nodes that regardless of distance has the smallest cost. In (a) it is clear that the right path is cheaper informationwise but longer in number of steps. (b) shows the fraction of minimum information paths that are also shortest paths in a scale-free network with $N=10^4$ nodes and $\gamma=2.4$. Although the overall fraction is as low as 0.62, S_{min} is only 5% smaller than S. As degeneracy is not considered in S_{min} we compare with S without degeneracy, S_u as in Fig. 3(a).

ity $(2/N)[1/(N/2-1)]$. But there are in fact $N/2$ degenerate paths, and the total information cost for locating parallel roads in this square city reduces to

$$S(\mathrm{NS} \rightarrow \mathrm{NS}) = -\log_2\left(\frac{N}{2}\frac{1}{N/2}\frac{1}{N/2-1}\right) = \log_2(N/2-1), \tag{13}$$

reflecting the fact that it does not matter which of the EW roads one uses to reach the target road. This places the fully connected bipartite network in the same class as the star network.

As an example of typical organization in social systems we also show the N dependence of modular networks in Fig. 7 [28]. Again, S is larger than in any random network irrespective of the degree distribution. We can therefore extend the previous statement that the value of $s=S/\log_2(N)$ is related to the global organization principle to include both the degree distribution, Fig. 5, and the way the nodes are positioned relatively to each other, Fig. 7.

Again, all results are robust to the details in the formulation of S and very similar results would have been obtained if we instead had considered S_u and excluded degeneracy or S_d with a different weight of the degenerate paths. To extend this we show in Fig. 8(b) the deviations between the search information S and a minimum search information S_{min}, where we take the minimum information concept to the extreme and look for the path *regardless of length* that has the smallest information cost. This would typically be a path that avoids hubs. In Fig. 8 it is obvious that the right choice is cheaper informationwise even though the path is longer. Intuitively the number of shortest information paths that also are shortest paths will decay as the paths get longer and longer. This is confirmed in Fig. 8(b) for a scale-free network of size $N=10^4$ with $\gamma=2.4$. Nevertheless, the difference from the shortest information path is small. This observation is valid in case of logarithmic information cost at every node, as in the present case. If the cost instead was linear as in Fig. 2(c), the difference would be substantial as hubs would repel the minimum information paths much more.

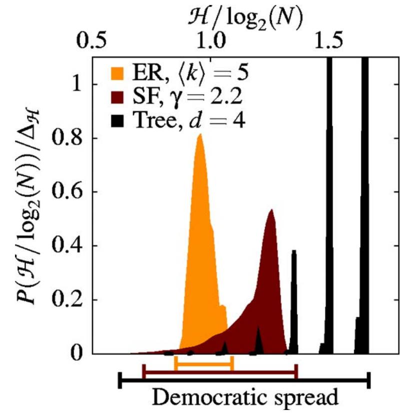


FIG. 9. (Color online) Distribution of hide $\mathcal{H}$ for nodes on various types of networks (low $\mathcal{H}$ means high visibility). We see that the Erdős-Rényi (ER) network is quite homogeneous, the scale-free (SF) network has a wider distribution, whereas the tree hierarchy is by far the least democratic in distributing the ability of different nodes to communicate. The democratic spread plotted below the box roughly estimates the division of communication in the network. The number of nodes is $N=10^4$ for all networks.

IV. NODE ORGANIZATION

We have until now presented a tool to characterize networks on the global level and quantified networks as being easy or difficult to navigate or search on average. We now turn to the effect the organization of networks has on the individual nodes. The specific communication approach opens up a natural way to characterize the different networks in terms of their ability to distribute communication options among their nodes. We therefore define the hide

$$\mathcal{H}_t = \sum_s S(s \rightarrow t)/N \tag{14}$$

as the average number of bits a walker needs to walk directly from a random node in the network to the target node t [12]. The different values of hide reflect to what degree the nodes are visible. Low hide $\mathcal{H}$, or low average information cost to find the node, represents high visibility. This is illustrated in Fig. 9, where in agreement with intuition we find that Erdős-Rényi networks are by far the most democratic, whereas scale-free and especially tree hierarchies are hugely elitist. In particular the tree hierarchy has localized all communication (low $\mathcal{H}$ means high visibility and thus ability to receive information) to the top nodes. In Fig. 9 we plot the democratic spread as the difference between the most and the least visible node in the network as an illustrative estimate of the distribution of communication in the network.

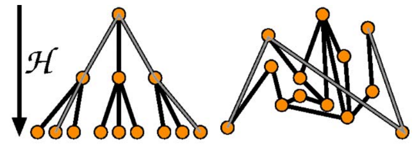
The different degrees of hide information of the various nodes effectively rank the nodes, and thereby suggest a self-consistent measure of a hierarchy based on visibility. At the same time the hide $\mathcal{H}$ captures both the hierarchy in the usual terms of trees, as in military structures, and the intrinsic hierarchical nature of topological hierarchies for scale-free networks [29] as in the Internet [16]. A highly ranked node is close to the top in a tree. The corresponding node in a topological hierarchy is a highly connected node. In the

Internet, for example, the highly connected nodes play the roles of intermediate nodes on typical paths between nodes further down in the hierarchy, just like top nodes in a tree. In analogy with [29,30] we define a path from s to t to be hierarchical if it defines a common boss for s and t. That is, the path has first to decrease monotonically in $\mathcal{H}_j$ to more and more visible nodes, until a minimum, and thereafter increase monotonically in $\mathcal{H}_j$ until the target node $j=t$ is reached. We allow the path to pass between nodes with the same value of $\mathcal{H}_j$, and we consider paths that only increase or only decrease as hierarchical. Given $\mathcal{H}_j$ for each node $j \in [1,N]$ in a network we quantify the network's degree of information hierarchy $\mathcal{F}_\mathcal{H}$ by the fraction of shortest paths between nodes in the network which are also hierarchical paths:

$$\mathcal{F}_\mathcal{H} = \frac{(\text{number of hierarchical shortest paths})}{N(N-1)}, \qquad (15)$$

where the denominator counts the total number of shortest paths between nodes in the network. In case of degenerate shortest paths, each path contributes to $\mathcal{F}_\mathcal{H}$ by a weight given by its contribution to the traffic betweenness. In accordance with intuition we find that $\mathcal{F}_\mathcal{H}$ decays with system size for random Erdős-Rényi networks, as shortest paths get longer. The decay is plotted in the inset of Fig. 10. $\mathcal{F}_\mathcal{H}=1$ for both hierarchies and club hierarchies whereas $\mathcal{F}_\mathcal{H}$ for random scale-free networks depends on degree distribution. Figure 10 shows how the information hierarchy varies with degree distribution for pure random scale-free networks parametrized by $P(k) \propto 1/k^\gamma$. As γ increases from 2 the network goes from being a complete information hierarchy with $\mathcal{F}_\mathcal{H}=1$ toward 0 when γ approaches 3, the average degree approaches 2, and shortest paths become long. For real-world networks the overall observation is that biological networks are antihierarchical with respect to $\mathcal{F}_\mathcal{H}$, while social and communication networks tend to be hierarchical (see table in Fig. 10). The Internet is a network of autonomous systems [31] that in this data set consists of 6474 nodes and 12 572 links and its degree distribution is scale-free with $P(k) \propto 1/k^{2.1}$. In the CEO network (6193 nodes and 43 074 links), chief executive officers are connected by links if they sit on the same board [32]. The city network is constructed by mapping 4127 streets, to nodes and 5565 intersections to links between the nodes in the Swedish city of Stockholm [5,33]. Fly is the protein interaction network in *Drosophilia melanogaster* detected by the two-hybrid experiment [34], and yeast refers to the similar network in *Saccharomyces cerevisiae* [35].

Overall, for scale-free networks, the information hierarchy $\mathcal{F}_\mathcal{H}$ quantitatively follows the topological hierarchy $\mathcal{F}$ presented in [29]. Thus networks with maximal (minimal) topological hierarchy $\mathcal{F}$ [29] also have large (small) $\mathcal{F}_\mathcal{H}$. But it is important that the information hierarchy allows for a natural generalization to non-scale-free networks, and is therefore a unified definition of hierarchical organization with the most visible node in the top. A less powerful ranking is the betweenness [24] as the betweenness is sensitive to links that shortcut important nodes. By adding links between the children of a top node as in the club tree in Fig. 7, the ranking changes completely as the betweenness for the top node in principle would be zero, whereas its position at the top would still be reflected by the hide ranking.



Network	N	$\mathcal{F}_\mathcal{H}$	$\mathcal{F}_\mathcal{H}^{\text{random}}$
Internet [30]	6474	0.80	0.81(1)
CEO [31]	6193	0.79	0.45(1)
City [32]	4127	0.34	0.24(1)
Fly [33]	2915	0.14	0.19(2)
Yeast [34]	921	0.12	0.21(4)

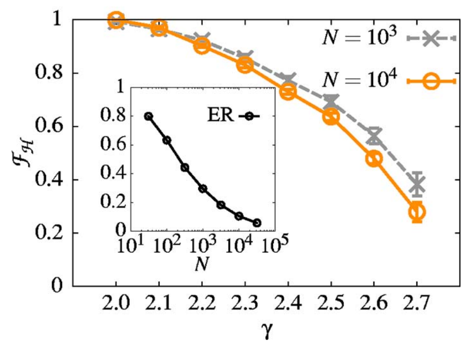


FIG. 10. (Color online) The hierarchical features of scale-free and Erdős-Rényi networks with hide $\mathcal{H}$ representing the importance of a node. The nodes in the networks in the top of the figure are arranged in hierarchical order. The intuitively hierarchical order of the tree is reproduced with the $\mathcal{H}$ ordering and all shortest paths are hierarchical, $\mathcal{F}_\mathcal{H}=1$. The network to the right has the opposite property. A high ratio of the shortest paths are not hierarchical since a typical path repeatedly goes up and down in the $\mathcal{H}$ hierarchy. The table shows a number of real-world networks and their $\mathcal{F}_\mathcal{H}$ together with the corresponding value in randomized networks with the same degree sequence [15,16]. The biological networks are randomized so that both bait and prey degree of all proteins are preserved. The plot in the bottom of the figure shows the behavior of scale-free and Erdős-Rényi networks with respect to $\mathcal{F}_\mathcal{H}$. Scale-free networks are γ dependent whereas Erdős-Rényi networks are size dependent.

V. CONCLUSION

Networks are a natural way to visualize the limited information access experienced by individual parts of the overall system. In the present paper we have explored topologies of a number of model networks in terms of their ability to facilitate peer-to-peer communication. The ability to transmit specific signals is quantified in terms of the difficulty in navigating the networks, quantified by the search information S. As an overall lesson we have found that the inequality

$$S(\text{real world}) > S(\text{random,fixed degree}) > S(\text{ER}), \qquad (16)$$

is valid for all investigated real-world networks [5,12–14]. Here S (random, fixed degree) represents randomized networks with exactly the same degree distribution as the investigated real-world network, whereas ER (Erdős-Rényi) networks only have the same total number of nodes and links as the real-world network. The above inequality is in particular associated with cases where the cost of passing a node is proportional to $\log(k)$, but it is also true for the higher local information cost proportional to k, where k is the degree of the node. As S represents an average of the contribution from any node to any other node, the major contribution to S comes from pairs of nodes that are separated by large distances. The fact that S in realistic networks is relatively large teaches us that the topology of real-world networks disfavors distant specific communication [13,14]. Topologically, large S was found in a number of model networks, with modular or hierarchical features with highly connected nodes deliberately positioned "between" other nodes, hinting that a large search information S is associated not only with broad degree distributions, but also with well known organizational features of social and biological systems.

The peer-to-peer search information $S(s \rightarrow t)$ opens the possibility for a detailed measure of the relative "importance" of nodes in a given network. In fact, measuring visibility of a node t in terms of how well hidden the node is from the rest of the network as in Eq. (14), we have shown how networks can be ranked in terms of a generalized hierarchy measure. The measure captures both the hierarchy in the usual terms of trees shown in Fig. 9 and at the same time also the intrinsic topological hierarchical nature of scale-free networks. Thus, this generalized hierarchy measure defines scale-free networks with degree distribution with exponent close to $\gamma = 2$ to be hierarchical, whereas narrower distributions will be antihierarchical unless they are deliberately organized in a treelike structure.

Overall, the different ways of organizing networks can be recast according to their ability or inability to transmit specific messages across the networks. The presented search information S provides a useful measure of this key functional role that is reflected in the topology of many real-world networks.

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